FINITE TYPE LINK-HOMOTOPY INVARIANTS

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ABSTRACT. An explicit polynomial in the linking numbers l_{ij} and Milnor's triple linking numbers $\mu(rst)$ on six component links is shown to be a well-defined finite type link-homotopy invariant. This solves a problem raised by B. Mellor and D. Thurston. An extension of our construction also produces a finite type link invariant which detects the invertibility for some links.

1. INTRODUCTION

The classification of links in the 3-space up to link-homotopy [2] was published ten years ago. Since then, the question of whether one could extract link-homotopy invariants from this classification has not been addressed properly. Recall that this classification starts with the classification of k component string links up to link-homotopy by a finitely generated torsion free nilpotent group $\mathcal{H}(k)$. Then link-homotopy classes are classified as orbits of this group $\mathcal{H}(k)$ under the "nilpotent action" of conjugations and partial conjugations. The group $\mathcal{H}(k)$ is of rank

$$\sum_{n=2}^{k} (n-2)! \binom{k}{n}$$

so an element of $\mathcal{H}(k)$ can be described uniquely by that many integers. These integers are Milnor's μ-numbers[1] with distinct indices. By a *link-homotopy invariant polynomial*, or simply a link-homotopy invariant, we mean a polynomial in these μ-numbers which is invariant under the action of conjugations and partial conjugations. There are trivial examples of such link-homotopy invariant polynomials coming from polynomials of linking numbers. A link-homotopy invariant polynomial is non-trivial if it contains higher order μ-numbers.

The main result of this paper is that such a non-trivial link-homotopy invariant polynomial exists when $k \geq 6$.

The abelianization of $\mathcal{H}(k)$ is a free abelian group of rank $\binom{k}{2}$. This is where the classical linking numbers l_{ij}, $1 \leq i < j \leq k$, fit in. The action of conjugations and partial conjugations on this quotient of $\mathcal{H}(k)$ is trivial. The next successive quotient of the lower central series of $\mathcal{H}(k)$ is a free abelian group of rank $\binom{k}{3}$, whose elements can be described by the collection of Milnor's triple linking number $\{\mu(rst); 1 \leq r < s < t \leq k\}$. The conjugations and partial conjugations act on this quotient by translations whose translation

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[1] Usually, they are called μ-invariants. But the word "invariant" is clearly abused here so we decide to call them μ-numbers.

vectors' coordinates are linear functions of the linking numbers l_{ij}. Thus, if the dimension of the subspace generated by these translation vectors is less then $\binom{k}{3}$ for generic values of the linking numbers, we may find a non-trivial vector perpendicular to all these translation vectors. Furthermore, the coordinates of this vector could be taken as polynomials in l_{ij}. Then the projection of an vector $\{\mu(rst)\}$ to this perpendicular vector will be invariant under conjugations and partial conjugations. This is the general philosophy behind our construction of link-homotopy invariant polynomials.

In a recent preprint [7], B. Mellor and D. Thurston has established the existence of link-homotopy invariants of finite type which are not polynomials of linking number when $k \geq 9$. Their proof is not constructive and therefore it is not clear whether their link-homotopy invariants are polynomials of μ-numbers.

On the other hand, since μ-numbers are of finite type for string links [4, 1], it is easy to see that our link-homotopy invariant polynomials are of finite type for links. For $k \leq 5$, it is shown in [7] that the only finite type link-homotopy invariants are polynomials in the linking numbers. So our construction fits nicely with this work of Mellor and Thurston.

Recall that the only finite type knot concordance invariant is the Arf invariant [8]. Since link concordance implies link-homotopy, our work (as well as the work of Mellor and Thurston, of course) shows the existence of non-trivial finite type link concordance invariants.

To extend the applicability of our general philosophy slightly, we find that the operation on the vector $\{\mu(rst)\}$ induced by reversing the orientation of each component of a string link is to change it by a negative sign followed by a translation whose translation vector's coordinates are quadratic polynomials in l_{ij}. If the dimension of the subspace generated by this vector together with the translation vectors of conjugations and partial conjugations is still less then $\binom{k}{3}$ for generic values of the linking numbers, and this is the case indeed, we can construct a non-trivial link-homotopy invariant polynomial which is changed by a sign when the orientation of each component of a link is reversed. We say that such a link invariant detects the invertibility for links. Recall that the reversion of the orientation of every component of a link does not change the quantum invariant associated with an irreducible representation of a semi-simple Lie algebra (see, for example, [5]). Thus our invariant is of finite type but is not determined by quantum invariants. The existence of a finite type knot invariant which detects the invertibility for knots is a major problem in the theory of finite type invariants (see, for example, [5, 3]). We believe that finite type knot invariants can not detect the invertibility for knots.

It remains unclear whether we can have a complete set of link-homotopy invariant polynomials which determines uniquely link-homotopy classes of links. See [6] for an earlier attempt on this problem. This problem could probably be translated to the problem of understanding the sublattice generated by the translation vectors of conjugations and partial conjugations. A better understanding of this sublattice might also be useful in answering the following question. If we let $\deg(l_{ij}) = 1$ and $\deg(\mu(rst)) = 2$, the link-homotopy invariant polynomial for $k = 6$ we construct in Section 3, which detects the invertibility for links, is a linear combination of 113,700 monomials of degree 22, homogeneous in both

l_{ij} and $\mu(rst)$ and linear in $\mu(rst)$. Is there a shorter link-homotopy invariant polynomial detecting the invertibility for links?

2. Conjugation and Partial Conjugation

We first recall the classification of ordered, oriented links up to link-homotopy given in [2]. We will follow the notations of [2].

Let $\mathcal{H}(k)$ be the group of link-homotopy classes of ordered, oriented string links with k components. The components of a string link will be ordered by $1, 2, \ldots, k$. It is shown in [2] that the pure braid group $P(k)$ of k components maps onto $\mathcal{H}(k)$ under the natural map $P(k) \rightarrow \mathcal{H}(k)$.

Deletion of the i-th component of the string link gives rise to a group homomorphism $d_i : \mathcal{H}(k) \rightarrow \mathcal{H}(k-1)$. If $F(k)$ denotes the free group of rank k generated by $x_1, x_2, \ldots, x_k$, the reduced free group $RF(k)$ is the quotient of $F(k)$ by adding relations $[x_i, x_i^g] = 1$ for all i and all $g \in F(k)$.

Lemma 2.1. *There is a split short exact sequence of groups*

(1) $$1 \rightarrow RF(k-1) \longrightarrow \mathcal{H}(k) \xrightarrow{d_i} \mathcal{H}(k-1) \rightarrow 1$$

where $RF(k-1)$ is the reduced free group generated by $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k$.

Notice that the split exact sequence (1) depends on the deleting component so that there are k such split exact sequences all together. A split exact sequence determines a semi-direct product decomposition
$$\mathcal{H}(k) = \mathcal{H}(k-1) \ltimes RF(k-1).$$

Conjugation in the group $\mathcal{H}(k)$ is defined as usual: A conjugation of $\sigma \in \mathcal{H}(k)$ by $\beta \in \mathcal{H}(k)$ is the element $\beta\sigma\beta^{-1} \in \mathcal{H}(k)$. A *partial conjugation* of $\sigma \in \mathcal{H}(k)$ is an element of the form $\theta h g h^{-1}$, where we write $\sigma = \theta g$ according to a decomposition $\mathcal{H}(k) = \mathcal{H}(k-1) \ltimes RF(k-1)$, for $\theta \in \mathcal{H}(k-1)$ and $g \in RF(k-1)$, and for an arbitrary $h \in RF(k-1)$.

It is easy to see that every link-homotopy class of ordered, oriented links with k components can be realized as the closure of an element in $\mathcal{H}(k)$. One of the main results of [2] is the following classification theorem.

Theorem 2.2. *Let $\sigma, \sigma' \in \mathcal{H}(k)$. Then the closures of σ and σ' are link-homotopic as ordered, oriented links if and only if there is a sequence $\sigma = \sigma_0, \sigma_1, \ldots, \sigma_n = \sigma'$ of elements of $\mathcal{H}(k)$ such that σ_{j+1} is either a conjugation or a partial conjugation of σ_j.*

For a group G, we will denote by G_n the n-th term of the lower central series of G, i.e. $G_1 = G$ and $G_{n+1} = [G_n, G]$, the normal subgroup of G generated by elements of the form $[g, h] = ghg^{-1}h^{-1}$ for all $g \in G_n$ and $h \in G$. A group G is nilpotent of class n if $G_{n+1} = 1$ but $G_n \neq 1$. We summarize some known facts about the group structures of $\mathcal{H}(k)$ in the following lemma.

Lemma 2.3. *1) $\mathcal{H}(k)$ is torsion free and nilpotent of class $k-1$;*

2) Corresponding to a decomposition $\mathcal{H}(k) = \mathcal{H}(k-1) \ltimes RF(k-1)$, we have
$$H(k)_n = \mathcal{H}(k-1)_n \ltimes RF(k-1)_n.$$

3) $\mathcal{H}(k)_{n-1}/\mathcal{H}(k)_n$ *is a free abelian group of rank* $(n-2)!\binom{k}{n}$.

For $\sigma \in \mathcal{H}(k)$, its image in $\mathcal{H}(k)/\mathcal{H}(k)_3$ can be described by $\binom{k}{2} + \binom{k}{3}$ integers. These integers are linking numbers l_{ij}, for $1 \le i < j \le k$, and Milnor's triple linking numbers $\mu(rst)$, for $1 \le r < s < t \le k$. We want to have them defined precisely and understand how they change when σ is changed by a conjugation or a partial conjugation.

We will denote by $\tau_{rs} = \tau_{sr}$, for $1 \le r < s \le k$, the pure braid depicted in Figure 1. Let $\sigma \in \mathcal{H}(k)/\mathcal{H}(k)_3$. For $1 \le r < s < t \le k$, after deleting all components other than the r, s, t-th components, σ can be written in the following normal form

$$(2) \qquad \sigma = \tau_{rs}^\alpha \tau_{rt}^\beta \tau_{st}^\gamma [\tau_{rt}, \tau_{st}]^\delta,$$

where $\alpha = l_{rs}$, $\beta = l_{rt}$, $\gamma = l_{st}$. By definition, we have $\delta = \mu(rst)$ for $\sigma \in \mathcal{H}(k)$.
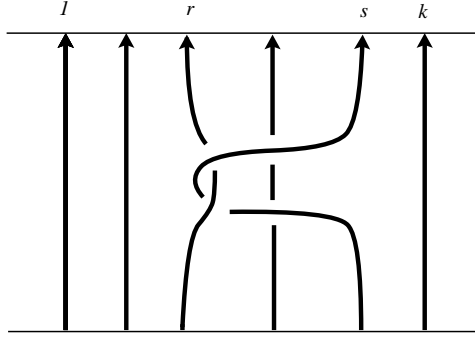

Figure 1. The pure braid τ_{rs}.

Lemma 2.4. *In* $\mathcal{H}(k)/\mathcal{H}(k)_3$, *if* r', s', t' *is a permutation of* r, s, t *and* ϵ *is the sign of the permutation, then*

$$[\tau_{r't'}, \tau_{s't'}] = [\tau_{rt}, \tau_{st}]^\epsilon.$$

Furthermore, we have

$$[\tau_{rt}^\eta, \tau_{st}] = [\tau_{rt}, \tau_{st}]^\eta.$$

This lemma is useful in the following calculation and its proof is straightforward.

To understand how $\mu(rst)$ changes under the conjugation, we only need to calculate the conjugation of $\sigma \in \mathcal{H}(k)/\mathcal{H}(k)_3$ under the normal form (2) by $\tau_{rs}, \tau_{rt}, \tau_{st}$. This calculation

is straightforward:

$$\tau_{rs}\sigma\tau_{rs}^{-1} = \tau_{rs}\tau_{rs}^{\alpha}\tau_{rt}^{\beta}\tau_{st}^{\gamma}[\tau_{rt},\tau_{st}]^{\delta}\tau_{rs}^{-1}$$

$$= \tau_{rs}^{\alpha}\tau_{rt}^{\beta}\tau_{st}^{\gamma}[\tau_{rs},\tau_{rt}]^{\beta}[\tau_{rs},\tau_{st}]^{\gamma}[\tau_{rt},\tau_{st}]^{\delta}$$

$$= \tau_{rs}^{\alpha}\tau_{rt}^{\beta}\tau_{st}^{\gamma}[\tau_{rt},\tau_{st}]^{\delta+\beta-\gamma};$$

$$\tau_{rt}\sigma\tau_{rt}^{-1} = \tau_{rt}\tau_{rs}^{\alpha}\tau_{rt}^{\beta}\tau_{st}^{\gamma}[\tau_{rt},\tau_{st}]^{\delta}\tau_{rt}^{-1}$$

$$= \tau_{rs}^{\alpha}\tau_{rt}^{\beta}\tau_{st}^{\gamma}[\tau_{rt},\tau_{rs}]^{\alpha}[\tau_{rt},\tau_{st}]^{\gamma}[\tau_{rt},\tau_{st}]^{\delta}$$

$$= \tau_{rs}^{\alpha}\tau_{rt}^{\beta}\tau_{st}^{\gamma}[\tau_{rt},\tau_{st}]^{\delta-\alpha+\gamma};$$

$$\tau_{st}\sigma\tau_{st}^{-1} = \tau_{st}\tau_{rs}^{\alpha}\tau_{rt}^{\beta}\tau_{st}^{\gamma}[\tau_{rt},\tau_{st}]^{\delta}\tau_{st}^{-1}$$

$$= \tau_{rs}^{\alpha}\tau_{rt}^{\beta}\tau_{st}^{\gamma}[\tau_{st},\tau_{rs}]^{\alpha}[\tau_{st},\tau_{rt}]^{\beta}[\tau_{rt},\tau_{st}]^{\delta}$$

$$= \tau_{rs}^{\alpha}\tau_{rt}^{\beta}\tau_{st}^{\gamma}[\tau_{rt},\tau_{st}]^{\delta+\alpha-\beta}.$$

We summarize the calculation into the following lemma.

Lemma 2.5. *The change of $\mu(rst)$ under a conjugation is given by*

$$\begin{aligned}
\textit{Conjugation by } \tau_{rs}: \quad & \mu(rst) \to \mu(rst) + l_{rt} - l_{st}; \\
\textit{Conjugation by } \tau_{rt}: \quad & \mu(rst) \to \mu(rst) - l_{rs} + l_{st}; \\
\textit{Conjugation by } \tau_{st}: \quad & \mu(rst) \to \mu(rst) + l_{rs} - l_{rt}.
\end{aligned}$$

Furthermore, $\mu(rst)$ will not change under a conjugation by τ_{ij} where $\{i,j\}$ and $\{r,s,t\}$ have at most one element in common.

The calculation of partial conjugations is slightly more complicated. We will start with partial conjugations by τ_{rt} and τ_{st}. These two operations are denoted by $\mathbf{t}^r$ and $\mathbf{t}^s$, respectively. For $\sigma \in \mathcal{H}(k)/\mathcal{H}(k)_3$ under the normal form (2), we have:

$$\sigma \xrightarrow{\ \mathbf{t}^r\ } \tau_{rs}^{\alpha}\tau_{rt}\tau_{rt}^{\beta}\tau_{st}^{\gamma}[\tau_{rt},\tau_{st}]^{\delta}\tau_{rt}^{-1}$$

$$= \tau_{rs}\alpha\tau_{rt}^{\beta}\tau_{st}^{\gamma}[\tau_{rt},\tau_{st}]^{\delta+\gamma};$$

$$\sigma \xrightarrow{\ \mathbf{t}^s\ } \tau_{rs}^{\alpha}\tau_{st}\tau_{rt}^{\beta}\tau_{st}^{\gamma}[\tau_{rt},\tau_{st}]^{\delta}\tau_{st}^{-1}$$

$$= \tau_{rs}\alpha\tau_{rt}^{\beta}\tau_{st}^{\gamma}[\tau_{rt},\tau_{st}]^{\delta-\beta}.$$

To calculate partial conjugations by τ_{rs} and τ_{ts}, which are denoted by $\mathbf{s}^r$ and $\mathbf{s}^t$, respectively, we need to rewrite σ as follows:

$$\sigma = \tau_{rs}^{\alpha}\tau_{rt}^{\beta}\tau_{st}^{\gamma}[\tau_{rt},\tau_{st}]^{\delta} = \tau_{rt}^{\beta}\tau_{rs}^{\alpha}\tau_{ts}^{\gamma}[\tau_{rs},\tau_{ts}]^{-\delta-\alpha\beta}.$$

Then, we have:

$$\sigma \xrightarrow{\mathbf{s}^r} \tau_{rt}^\beta \tau_{rs} \tau_{rs}^\alpha \tau_{ts}^\gamma [\tau_{rs}, \tau_{ts}]^{-\delta-\alpha\beta} \tau_{rs}^{-1}$$

$$= \tau_{rt}^\beta \tau_{rs}^\alpha \tau_{ts}^\gamma [\tau_{rs}, \tau_{ts}]^{-\delta-\alpha\beta+\gamma}$$

$$= \tau_{rs}^\alpha \tau_{rt}^\beta \tau_{st}^\gamma [\tau_{rt}, \tau_{st}]^{\delta-\gamma};$$

$$\sigma \xrightarrow{\mathbf{s}^t} \tau_{rt}^\beta \tau_{ts} \tau_{rs}^\alpha \tau_{ts}^\gamma [\tau_{rs}, \tau_{ts}]^{-\delta-\alpha\beta} \tau_{ts}^{-1}$$

$$= \tau_{rt}^\beta \tau_{rs}^\alpha \tau_{ts}^\gamma [\tau_{rs}, \tau_{ts}]^{-\delta-\alpha\beta-\alpha}$$

$$= \tau_{rs}^\alpha \tau_{rt}^\beta \tau_{st}^\gamma [\tau_{rt}, \tau_{st}]^{\delta+\gamma}.$$

Similarly, to calculate partial conjugations $\mathbf{r}^s$ and $\mathbf{r}^t$, we first rewrite σ:

$$\sigma = \tau_{rs}^\alpha \tau_{rt}^\beta \tau_{st}^\gamma [\tau_{rt}, \tau_{st}]^\delta = \tau_{st}^\gamma \tau_{sr}^\alpha \tau_{tr}^\beta [\tau_{sr}, \tau_{tr}]^{\delta-\alpha\gamma+\beta\gamma}.$$

Then, we have

$$\sigma \xrightarrow{\mathbf{r}^s} \tau_{st}^\gamma \tau_{sr} \tau_{sr}^\alpha \tau_{tr}^\beta [\tau_{sr}, \tau_{tr}]^{\delta-\alpha\gamma+\beta\gamma} \tau_{sr}^{-1}$$

$$= \tau_{st}^\gamma \tau_{sr}^\alpha \tau_{tr}^\beta [\tau_{sr}, \tau_{tr}]^{\delta-\alpha\gamma+\beta\gamma+\beta}$$

$$= \tau_{rs}^\alpha \tau_{rt}^\beta \tau_{st}^\gamma [\tau_{rt}, \tau_{st}]^{\delta+\beta};$$

$$\sigma \xrightarrow{\mathbf{r}^t} \tau_{st}^\gamma \tau_{tr} \tau_{sr}^\alpha \tau_{tr}^\beta [\tau_{sr}, \tau_{tr}]^{\delta-\alpha\gamma+\beta\gamma} \tau_{tr}^{-1}$$

$$= \tau_{st}^\gamma \tau_{sr}^\alpha \tau_{tr}^\beta [\tau_{sr}, \tau_{tr}]^{\delta-\alpha\gamma+\beta\gamma-\alpha}$$

$$= \tau_{rs}^\alpha \tau_{rt}^\beta \tau_{st}^\gamma [\tau_{rt}, \tau_{st}]^{\delta-\alpha}.$$

We summarize the previous calculation into the following lemma.

Lemma 2.6. *The change of $\mu(rst)$ under a partial conjugation is given by*

$$\mathbf{t}^r : \quad \mu(rst) \to \mu(rst) + l_{st};$$
$$\mathbf{t}^s : \quad \mu(rst) \to \mu(rst) - l_{rt};$$
$$\mathbf{s}^r : \quad \mu(rst) \to \mu(rst) - l_{st};$$
$$\mathbf{s}^t : \quad \mu(rst) \to \mu(rst) + l_{rs};$$
$$\mathbf{r}^s : \quad \mu(rst) \to \mu(rst) + l_{rt};$$
$$\mathbf{r}^t : \quad \mu(rst) \to \mu(rst) - l_{rs}.$$

Furthermore, a partial conjugation by $\mathbf{i}^j$ will not change $\mu(rst)$ if $\{i, j\}$ and $\{r, s, t\}$ have at most one element in common.

For a given string link $\sigma \in \mathcal{H}(k)$, we will think of the whole collection $\{\mu(rst)\,;\, 1 \le r < s < t \le k\}$ as an element in $\mathbb{Z}^{\binom{k}{3}}$. Then the conjugations and partial conjugations act on $\mathbb{Z}^{\binom{k}{3}}$ by translations. We will abuse the notation by using the same symbol to denote both a translation operation and the corresponding translation vector. Thus, a translation

operation $T : V \to V$ on a vector space V is given by $T(v) = v + T$, for all $v \in V$ and a fixed $T \in V$. If T_1 and T_2 are two translations, we have

$$(T_1 \cdot T_2)(v) = v + T_1 + T_2, \qquad \text{for all } v \in V.$$

The following two theorems follow directly from Lemmas 2.5 and 2.6.

Theorem 2.7. *The translation operation on $\mathbb{Z}^{\binom{k}{3}}$ given by the conjugation of τ_{ij} is the same as the composition of the translation operations given by the partial conjugations $\mathbf{i}^j$ and $\mathbf{j}^i$, i.e. it is equal to $\mathbf{i}^j + \mathbf{j}^i$.*

Theorem 2.8. *The translation operations $\mathbf{i}^j$ satisfy the following relations:*

$$\sum_{j \neq i} \mathbf{j}^i = 0$$

$$\sum_{j \neq i} l_{ij}\, \mathbf{i}^j = 0$$

for all $i = 1, 2, \ldots, k$.

String links are oriented in the sense that each component is given a orientation from the bottom to the top. See Figure 1. Reversing the orientation on each component of a string link defines bijection

$$\sigma \mapsto \overline{\sigma} : \mathcal{H}(k) \to \mathcal{H}(k).$$

This bijection is an anti-homomorphism: $\overline{\sigma_1 \sigma_2} = \overline{\sigma_2}\,\overline{\sigma_1}$. This bijection induces an operation on $\mathbb{Z}^{\binom{k}{3}}$.

Theorem 2.9. *The operation on $\mathbb{Z}^{\binom{k}{3}}$ induced by reversing the orientation of each component of a string link is to change each $\mu(rst)$ to $-\mu(rst)$ followed by the translation operation*

$$\mu(rst) \longrightarrow \mu(rst) - l_{rs}\, l_{rt} + l_{rs}\, l_{st} - l_{rt}\, l_{st}.$$

Proof. Consider the normal form (2) of $\sigma \in \mathcal{H}(k)/\mathcal{H}(k)_3$ in the r, s, t-th components. The normal form for $\overline{\sigma}$ is obtained as follows:

$$\overline{\sigma} = [\tau_{rt}, \tau_{st}]^{-\delta} \tau_{st}^{\gamma} \tau_{rt}^{\beta} \tau_{rs}^{\alpha}$$
$$= \tau_{rs}^{\alpha} \tau_{rt}^{\beta} \tau_{st}^{\gamma} [\tau_{rt}, \tau_{st}]^{-\delta - \alpha\beta + \alpha\gamma - \beta\gamma}.$$

Thus the operation on $Z^{\binom{k}{3}}$ induced by $\sigma \mapsto \bar{\sigma}$ is given by

$$\mu(rst) \longrightarrow -\mu(rst) - l_{rs}\, l_{rt} + l_{rs}\, l_{st} - l_{rt}\, l_{st}.$$

$\square$

3. Construction of the Invariant

By Theorems 2.2 and 2.7, we shall look for polynomials in l_{ij} and $\mu(rst)$ invariant under the translation operations on $\{\mu(rst)\} \in \mathbb{Z}^{\binom{k}{3}}$ induced by partial conjugations. There are $k(k-1)$ partial conjugations all together and their induced translations subject to $2k$ linear equations given in Theorem 2.8. If these equations are linearly independent for generic values of $\{l_{ij}\}$, the sublattice of $Z^{\binom{k}{3}}$ generated by the translation vectors of the partial conjugations will be of dimension no larger than $k(k-1) - 2k = k^2 - 3k$.

Lemma 3.1. *For $k > 3$, the $2k$ equations in Theorem 2.8 are linearly independent for generic values of $\{l_{ij}\}$.*

Proof. We write the two sets of equations in Theorem 2.8 as follows:
$$\mathbf{1}^i + \mathbf{2}^i + \cdots + \mathbf{j}^i + \cdots + \mathbf{k}^i = 0, \quad j \neq i;$$
$$l_{i1}\mathbf{i}^1 + l_{i2}\mathbf{i}^2 + \cdots + l_{ij}\mathbf{i}^j + \cdots + l_{ik}\mathbf{i}^k = 0, \quad j \neq i,$$
for each $i = 1, 2, \ldots, k$.

For generic values of $\{l_{ij}\}$, use the first $k-1$ equations the first set of k equations, we can solve for $\mathbf{k}^1, \mathbf{k}^2, \ldots, \mathbf{k}^{k-1}$. Similarly, we can solve for $\mathbf{1}^k, \mathbf{2}^k, \ldots, (\mathbf{k-1})^k$ from the first $k-1$ equations of the second set of k equations. The remaining vectors $\mathbf{i}^j$, $i, j \neq k$, have to satisfy another two equations obtained from the last equations in those two sets of k equations, respectively, by substituting $\mathbf{k}^i$ and $\mathbf{i}^k$ with their solutions in terms of $\mathbf{i}^j$ for $i, j \neq k$. It it then easy to check that these two equations are linearly independent when $k > 3$. $\square$

Lemma 3.2. *For $k = 4, 5$, we have $\binom{k}{3} = k^2 - 3k$. For $k \geq 6$, we have $\binom{k}{3} > k^2 - 3k$.*

Proof. The case of $k = 4, 5$ can be checked directly. For $k \geq 6$, we have
$$\binom{k}{3} - (k^2 - 3k) = \frac{k}{6}(k^2 - 9k + 20)$$
$$= \frac{k}{6}((k-6)^2 + 3k - 16) > 0.$$
$\square$

Theorem 3.3. *For $k \geq 6$, there exists a polynomial in l_{ij} and $\mu(rst)$ which is a link-homotopy invariant of ordered, oriented links with k components. This link-homotopy invariant is of finite type.*

Proof. In $\mathbb{Z}^{\binom{k}{3}}$, let $\mathcal{P}$ be the sublattice generated by the translation vectors of partial conjugations. Then we have
$$\dim(\mathcal{P}) \leq k^2 - 3k < \binom{k}{3}.$$

Let $\Omega \in \mathbb{Z}^{\binom{k}{3}}$ be a non-zero vector perpendicular to $\mathcal{P}$. We can choose such a Ω so that its coordinates are polynomials in $\{l_{ij}\}$ and the inner product $\mathbf{i}^j \cdot \Omega$ is identically zero. This can be achieved by considering generic values of $\{l_{ij}\}$ first. Then since $\mathbf{i}^j \cdot \Omega = 0$ for generic

values of $\{l_{ij}\}$, it has to be zero identically. Let $\mu = \{\mu(rst)\} \in \mathbb{Z}^{\binom{k}{3}}$. The inner product $\mu \cdot \Omega$ is invariant under the translations by vectors in $\mathcal{P}$. This is a desired link-homotopy invariant of ordered, oriented links since

$$(\mu + \mathbf{i}^j) \cdot \Omega = \mu \cdot \Omega$$

for all $i, j = 1, 2, \ldots, k$.

The fact that the invariant $\mu \cdot \Omega$ is of fine type is a direct consequence of the fact that the linking numbers and the triple linking numbers are all finite type invariants of string links ([4, 1]). If we have a singular link, we may put it into the form of the closure of a single string link. Since polynomials of finite type invariants are still of finite type, $\mu \cdot \Omega$ vanishes on singular string links with sufficiently large number of double points. This implies that it is a finite type link invariant. $\qquad\square$

We now consider in some details the case $k = 6$.

Let us order $\mu(rst)$, $1 \le r < s < t \le 6$ in the lexicographic order. So

$$\mu = (\mu(123), \mu(124), \mu(125), \mu(126), \mu(134), \mu(135), \mu(136), \mu(145), \mu(146), \mu(156),$$
$$\mu(234), \mu(235), \mu(236), \mu(245), \mu(246), \mu(256), \mu(345), \mu(346), \mu(356), \mu(456)).$$

Then the vectors of the translation operations $\mathbf{1}^2, \mathbf{1}^3, \mathbf{1}^4, \mathbf{1}^5, \mathbf{1}^6, \mathbf{2}^1, \mathbf{2}^3, \mathbf{2}^4, \mathbf{2}^5, \mathbf{2}^6, \mathbf{3}^1, \mathbf{3}^2,$ $\mathbf{3}^4, \mathbf{3}^5, \mathbf{3}^6, \mathbf{4}^1, \mathbf{4}^2, \mathbf{4}^3, \mathbf{4}^5, \mathbf{4}^6, \mathbf{5}^1, \mathbf{5}^2, \mathbf{5}^3, \mathbf{5}^4, \mathbf{5}^6, \mathbf{6}^1, \mathbf{6}^2, \mathbf{6}^3, \mathbf{6}^4, \mathbf{6}^5$ are the row vectors of the following 30×20 matrix from the top to the bottom, respectively:

$$
\begin{vmatrix}
l_{13} & l_{14} & l_{15} & l_{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-l_{12} & 0 & 0 & 0 & l_{14} & l_{15} & l_{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -l_{12} & 0 & 0 & -l_{13} & 0 & 0 & l_{15} & l_{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -l_{12} & 0 & 0 & -l_{13} & 0 & -l_{14} & 0 & l_{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -l_{12} & 0 & 0 & -l_{13} & 0 & -L_{14} & -l_{15} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-l_{23} & -l_{24} & -l_{25} & -l_{26} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
l_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & l_{24} & l_{25} & l_{26} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & l_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -l_{23} & 0 & 0 & l_{25} & l_{26} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & l_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -l_{23} & 0 & -l_{24} & 0 & l_{26} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & l_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -l_{23} & 0 & -l_{24} & -l_{25} & 0 & 0 & 0 & 0 \\
l_{23} & 0 & 0 & 0 & -l_{34} & -l_{35} & -l_{36} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-l_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -l_{34} & -l_{35} & -l_{36} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & l_{13} & 0 & 0 & 0 & 0 & 0 & l_{23} & 0 & 0 & 0 & 0 & 0 & l_{35} & l_{36} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & l_{13} & 0 & 0 & 0 & 0 & 0 & l_{23} & 0 & 0 & 0 & 0 & -l_{34} & 0 & l_{36} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & l_{13} & 0 & 0 & 0 & 0 & 0 & l_{23} & 0 & 0 & 0 & 0 & -l_{34} & -l_{35} & 0 \\
0 & l_{24} & 0 & 0 & l_{34} & 0 & 0 & -l_{45} & -l_{46} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -l_{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & l_{34} & 0 & 0 & -l_{45} & -l_{46} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -l_{14} & 0 & 0 & 0 & 0 & 0 & -l_{24} & 0 & 0 & 0 & 0 & 0 & -l_{45} & -l_{46} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & l_{14} & 0 & 0 & 0 & 0 & 0 & l_{24} & 0 & 0 & l_{34} & 0 & 0 & l_{46} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & l_{14} & 0 & 0 & 0 & 0 & 0 & l_{24} & 0 & 0 & l_{34} & 0 & -l_{45} \\
0 & 0 & l_{25} & 0 & 0 & l_{35} & 0 & l_{45} & 0 & -l_{56} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -l_{15} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & l_{35} & 0 & l_{45} & 0 & -l_{56} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -l_{15} & 0 & 0 & 0 & 0 & 0 & -l_{25} & 0 & 0 & 0 & 0 & l_{45} & 0 & -l_{56} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -l_{15} & 0 & 0 & 0 & 0 & 0 & -l_{25} & 0 & 0 & -l_{35} & 0 & 0 & -l_{56} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & l_{15} & 0 & 0 & 0 & 0 & 0 & 0 & l_{25} & 0 & l_{35} & l_{45} \\
0 & 0 & 0 & l_{26} & 0 & 0 & l_{36} & 0 & l_{46} & l_{56} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -l_{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & l_{36} & 0 & l_{46} & l_{56} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -l_{16} & 0 & 0 & 0 & 0 & 0 & -l_{26} & 0 & 0 & 0 & 0 & l_{46} & l_{56} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -l_{16} & 0 & 0 & 0 & 0 & 0 & -l_{26} & 0 & 0 & -l_{36} & 0 & l_{56} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -l_{16} & 0 & 0 & 0 & 0 & 0 & l_{26} & 0 & 0 & -l_{36} & -l_{46}
\end{vmatrix}
$$

We shall pick out the 18 rows of this matrix corresponding to the translation operations of $\mathbf{1}^2$, $\mathbf{1}^3$, $\mathbf{1}^4$, $\mathbf{1}^5$, $\mathbf{2}^1$, $\mathbf{2}^3$, $\mathbf{2}^4$, $\mathbf{2}^5$, $\mathbf{3}^1$, $\mathbf{3}^2$, $\mathbf{3}^4$, $\mathbf{3}^5$, $\mathbf{4}^1$, $\mathbf{4}^2$, $\mathbf{4}^3$, $\mathbf{4}^5$, $\mathbf{5}^1$, $\mathbf{5}^2$, respectively. Calculation using Mathematica shows that these 18 vectors are linearly independent generically.

Consider now the operation of reversing the orientation. The vector $R = \{R(rst)\} \in \mathbb{Z}^{20}$ of the translation operation in Theorem 2.9 is given by

$$R(rst) = -l_{rs}\, l_{rt} + l_{rs}\, l_{st} - l_{rt}\, l_{st}.$$

One can verify that the vector R and the previous 18 vectors are linearly independent. Let $\mathcal{M}$ be the 19×20 matrix formed by these 19 vectors. Let $\mathcal{M}^{(i)}$ be the 19×19 matrix obtained from $\mathcal{M}$ by deleting the i-th column from $\mathcal{M}$, $i = 1, 2, \ldots, 20$. Let

$$\Omega_i = (-1)^{i-1}\det\left(\mathcal{M}^{(i)}\right)$$

and $\Omega = (\Omega_1, \Omega_2, \ldots, \Omega_{20})$.

Theorem 3.4. *$\mu \cdot \Omega$ is a finite type link-homotopy invariant of ordered, oriented links with 6 component. When the orientation of every component is reversed, this invariant is changed only by a sign.*

Proof. Using the fact that the rows of the cofactor matrix A^* of a given matrix A are perpendicular to different rows of A, we see that Ω is perpendicular to all the vectors of translation operation induced by partial conjugations as well as the vector R. Certainly, $\Omega \neq 0$. So $\mu \cdot \Omega$ is a non-trivial link-homotopy invariant of ordered, oriented links with 6 components. It is of finite type since it is a polynomial in l_{ij} and $\mu(rst)$. Under the reversion of orientation, μ changes to $-\mu + R$. Since $R \cdot \Omega = 0$, the invariant $\mu \cdot \Omega$ is only changed by a sign under the reversion of orientation. $\square$

To finish, let us furnish some data obtained using Mathematica. Let $\deg(l_{ij}) = 1$, then Ω_i is a homogeneous polynomial of degree 20 in l_{ij}. Let L_i be the number of monomials in Ω_i, the sequence $\{L_1, L_2, \ldots, L_{20}\}$ is given as follows:

$$\{5531, 5555, 5555, 5531, 5424, 5769, 5802, 5734, 5753, 5432,$$
$$5432, 5753, 5802, 5734, 5769, 5424, 5928, 5922, 5922, 5928\}.$$

Thus $\mu \cdot \Omega$ is linear and homogeneous in $\mu(rst)$ and has 113,700 monomials.

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